POVERTY RANKINGS OF OPPORTUNITY PROFILES

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1. Introduction

Poverty reduction plays a prominent role in political debates in many countries. Methods and techniques to make poverty comparisons are necessary tools in order to design and to evaluate policies aimed at poverty reduction.

Since the publication of Sen's (1976) pioneering paper on poverty measurement, in the last quarter century a great deal has been written on this subject. Several measures of poverty, including the one suggested by Sen (1976), are now available in the literature. However, in most of the existing literature, income or consumption expenditures has been regarded as the only relevant dimension of poverty. But poverty is essentially a multidimensional phenomenon and the exclusive reliance on just one indicator can hide crucial aspects of economic deprivation.

For example, two societies with the same distribution of monetary earnings, can hardly be thought of as equivalent in terms of poverty if in one of them fractions of the population are denied a number of basic rights and liberties such as the right to vote, freedom of speech, freedom of movement and so on.

The necessity to move from an income-based evaluation of social inequities towards a more comprehensive domain has been defended, among others, in the influential works of Rawls (1971), Sen (1980, 1997), Roemer (1996).

Altought the inadequacy of a unidimensional approach to evaluating social inequities is well recognized, it is nevertheless a common practice of economists to do so. One of the basic reasons for this is linked to the difficulties in data collection and data analysis . However, in addition to data limitation and empirical constraints, a multidimensonal evaluation of poverty is by no means straightforward from a theoretical point of view. Little is known about how to compare different distributions of, say, rights, freedoms, primary goods, or functionings. The focus of the present work is, in fact, this particular measurement problem.

In this paper, we consider the problem of ranking distributions of opportunities on the basis of poverty. An individual's opportunities are described by a set rather than by a scalar, as it is the case with income or consumption. As a consequence, the problem becomes that of ranking different distributions of opportunity sets.

Date: June 11, 2008.

To keep the approach as general as possible, the notion of "opportunity" is treated in an abstract way: we define an opportunity set as any finite set in some arbitrary space[1]. Opportunities may be thought of as non welfare characteristics of agents such as basic liberties, political rights, and individual freedoms; or as access to certain welfare enhancing traits; a further intepretation is in terms of functioning à la Sen (such as being educated, being well-nourished, avoiding premature mortality): in this case the opportunity set corresponds to the capability set of an individual.

The question of how to rank different opportunity distributions has been first addressed by Kranich (1996), who however focused only on inequality rankings. There is now an extensive literature concerned with the measurement of inequality of opportunity: see, for example, Arlegi and Nieto [1], Bossert, Fleurbaey, and Van de gaer [6], Herrero [10], Herrero, Iturbe-Ormaetxe, and Nieto [11], Kranich [14, 15], Ok [16], Ok and Kranich [17], and Savaglio and Vannucci [20]. A survey of this literature may be found in Barbera' et al. [3].

On the other hand, the question of how to rank different distributions of opportunities in terms of the poverty they exhibit has never been addressed before. The present paper fills this gap. We address the problem of ranking profiles of opportunity sets on the basis of poverty.

A natural approach towards devising a poverty ranking for opportunity distributions is to try to extend the basic income poverty measures into our richer setting. We, therefore, design our analysis by studying alternative ways of extending the familiar notion of "poverty line" and the most well known poverty measures in the context of opportunity distributions. In order to identify the different value systems involved in the use of different poverty criteria we use the axiomatic metodhology: we propose a number of properties that a poverty relation on the possible distributions (profiles) of finite opportunity sets should satisfy and we study their logical implications. We characterize two fundamental rankings: the Head-Count and the Opportunity-Gap poverty rankings. These generalize the most widely used poverty measures used in the income poverty framework, namely the *head count ratio* and *the income poverty gap*. In addition, we characterize axiomatically two lexicographic rankings based on the HC and OG rankings and a third one based on a linear combination of the head-count and gap criteria.

The paper is organized as follow. The next section introduces the analitical setting and defines formally the basic problem studied in this paper. Section 3 introduces and discuss a first set of axioms. Section 4 contains the main results of the paper: the characterization of the the Head-Count and the Opportunity-Gap poverty rankings. Section 5 contains additional sets of axioms and the characterization of composite rankings based on the HC and OG. Section 6 concludes with a brief discussion of the results and of directions for future research.

[1]This fundamental modeling choice makes our study different (and possibly more general) than the existing literature on multidimensional poverty mesurement: see, among others,. Tsui (2002), Chakravarty et al. (1998) and Bourguignon and Chakravarty (1999, 2002).

2. The framework

We start by identifying a universal nonempty set of opportunities, denoted by X. We assume that each element in X is desirable in some universal sense. Moreover, following the existing literature, we assume that opportunities are nonrival, so that a given opportunity is potentially available to everyone simultaneously, and that opportunities are excludable, so that providing an opportunity to some individuals does not necessarily imply that everyone has this opportunity.

Let $N = \{1, .., n\}$ denote the finite set of relevant population units and $\mathcal{P}[X]$ the set of all *finite* subsets of X. Elements of $\mathcal{P}[X]$ are referred to as *opportunity sets*, and mappings $\mathbf{Y} = (Y_1, ..., Y_n) \in \mathcal{P}[X]^N$ as profiles of opportunity sets, or simply *opportunity profiles.*

Hence, each individual in a society is endowed with an *opportunity set* and a society is represented by an *opportunity profiles.* We are interested in ranking such opportunity profiles in terms of poverty.

Folwing Sen's approach, the evaluation of poverty can be divided into two steps: (i) the identification step, in which the poor are identified in a given society; (ii) the aggregation step, in which the characteristics of the poor have to be aggregated in order to obtain an assesement of the aggregate poverty in a society.

As for the identification step, in the unidimensional context an income poverty line is chosen that divides the population into two sets: the poor and the non-poor. The identification of that income level below which people are described as poor can follow an absolute or a relative approach. While with an absolute approach the poverty line is defined in an exogenous way and is the same across distributions, with a relative approach the poverty line in a distribution is a function of the distribution (e.g. the poverty line can be fixed at half the median income level in that society).

An analog of the poverty line in our framework is a *poverty threshold,* which is a set $T \in \mathcal{P}[X]$. The poverty threshold T identifies a set of *essential alternatives*: an individual is declared as poor or, equivalently, he is declared to be *below* the poverty treshold, if her set does not contain all the essential alternatives, i.e., all the altarnatives contained in T. The set T is not dependent on the specific profile: i.e., in the identification step we adopt an absolute approach.

One possible interpretation of such essential alternatives is linked to the basic needs approach: hence having access to alla essential alternatives in thi ntepretation means being able to satisfy all basic needs. A related interpretation is related to Sen's framework: in such a case the essential alternatives represent certain basic functionings such as, for example, life expectancy, literacy, and so on.

As we shall see, the identification of essential alternatives and the distinction between essential and non essential alternatives plays a crucial role in our axiomatic cosntruction.

As for the aggregation step, the problem is that of "putting together" the information on the deprivation suffered by the poor in a society, to arrive to a unique evaluation of the aggregate poverty.

As we are working in a multidimensional domain of opportunites, our first step involves the definition of a metric in the space of opportunity sets: in other words, we need first to define a criterion to compare different individuals endowed with different opportunity sets. When is one person poorer (richer) than onother person? There is an extensive literature devoted to the problem of ranking opportunity sets (see on this the excelent survey by Barberà, Bossert and Xu, 2001). We propose a very mild condition: according to our criterion all the sets above the poverty treshold are each other indifferent; as for the sets below the poverty tresholds, they are ranked by set inclusion. Therefore, we consider a unique indifference class within the universe of the non–poor and we propose set inclusion as the reference ranking rule within the poor.

Formally, we introduce a preorder $\succcurlyeq_T^*$on $\mathcal{P}[X]$ which is induced by the poverty treshold T.

Definition 1. *For any $Y, Z \in \mathcal{P}[X]$, $Y \succcurlyeq_T^* Z$ iff [$Y \supseteq Z$ or $Y \supseteq T$].*

The notation $\mathbf{Y}_{|T}$ will be employed in the rest of this paper to denote opportunity profile $(Y_i \cap T)_{i \in N}$.

A *poverty ranking* of opportunity profiles -under threshold T- is a preorder $\succcurlyeq_T$ on $\mathcal{P}[X]^N$ such that for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$ whenever $Z_i \succcurlyeq_T^* Y_i$ for each $i \in N$. In the next section we shall propose some desirable properties that a poverty ranking should satisfy. The axiomatic structure will lead us to the characterization of different poverty criteria, which are based on generalization of the most widely used income poverty measures, namely the *head count ratio* and the *income poverty gap*.

Definition 2. *The* head-count (HC) poverty ranking *-under threshold T- is the preorder $\succcurlyeq_T^h$ on $\mathcal{P}[X]^N$ defined as follows: for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, $\mathbf{Y} \succcurlyeq_T^h \mathbf{Z}$ iff $h_T(\mathbf{Y}) \geq h_T(\mathbf{Z})$ where for each $\mathbf{W} \in \mathcal{P}[X]^N$, $h_T(\mathbf{W}) = \# H_T(\mathbf{W})$, and $H_T(\mathbf{W}) = \{i \in N : W_i \nsupseteq T\}$.*

The head-count poverty ordering ranks two distributions on the basis of the number of individuals that are below the poverty treshold T; hence, it captures the incidence of poverty. Althought such measures gives useful information on the poverty in a distribution, the head-count does not take into account the depth or the severity of the deprivation sufferend by the poor. In order to capture this aspect of the aggregate poverty, we also propose the *opportunity-gap (OG) poverty ranking* which measures the aggregate intensity of poverty.

Definition 3. *The* opportunity-gap (OG) poverty ranking *-under threshold T- is the preorder $\succcurlyeq_T^g$ on $\mathcal{P}[X]^N$ defined as follows: for any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, $\mathbf{Y} \succcurlyeq_T^h \mathbf{Z}$ iff $g_T(\mathbf{Y}) \geq g_T(\mathbf{Z})$, where for each $\mathbf{W} \in \mathcal{P}[X]^N$, $g_T(\mathbf{W}) = \sum_{i \in H_T(\mathbf{W})} \# \{x : x \in T \setminus W_i\}$.*

Thus, for each poor individual, the intensity of poverty - the "individual gap" - is measured by the number of essential alternatives he does not have access to. The opportunity-gap poverty ranking aggregates this information by summing the individual gaps; hence, it tells us how poor are the poor.

3. The axioms

We introduce now the following basic properties for a poverty ranking $\succcurlyeq_T$of $\mathcal{P}[X]^N$.

Axiom 1 (*Anonymity (AN)*)**.** *For any permutation π of N, and any $\mathbf{Y} \in \mathcal{P}[X]^N$: $\mathbf{Y} \sim_T \pi\mathbf{Y}$ (where $\pi\mathbf{Y} = (Y_{\pi(1)}, ..., Y_{\pi(n)})$).*

Axiom 2 (*Irrelevance of Inessential Opportunities (IIO)*)**.** *For any $\mathbf{Y} \in \mathcal{P}[X]^N$, $i \in N$, and $x \in Y_i \setminus T$: $\mathbf{Y} \sim_T (\mathbf{Y}_{-i}, Y_i \setminus \{x\})$.*

Axiom 3 (*Irrelevance of Poor's Opportunity Deletions (IPOD)*)**.** *For any $\mathbf{Y} \in \mathcal{P}[X]^N$, $i \in H_T(\mathbf{Y})$, and $x \in Y_i$: $\mathbf{Y} \sim_T (\mathbf{Y}_{-i}, Y_i \setminus \{x\})$.*

Axiom 4 (*Dominance at Essential Profiles (DEP)*)**.** *For any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that both $\{Y_1, ..., Y_n\} \subseteq \{T, \emptyset\}$ and $\{Z_1, ..., Z_n\} \subseteq \{T, \emptyset\}$, $\mathbf{Y} \succ_T \mathbf{Z}$ iff $\# \{i \in N : Y_i = \emptyset\} > \# \{i \in N : Z_i = \emptyset\}$.*

Axiom 5 (*Strict Monotonicity with respect to Essential Deletions (SMED)*)**.** *For any $\mathbf{Y} \in \mathcal{P}[X]^N$, $i \in N$, and $x \in Y_i \cap T$: $(\mathbf{Y}_{-i}, Y_i \setminus \{x\}) \succ_T \mathbf{Y}$.*

Axiom 6 (*Independence of Balanced Essential Deletions (IBED)*)**.** *For any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, $i \in N$, $y \in Y_i \cap T$ and $z \in Z_i \cap T$: $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$ iff $(\mathbf{Y}_{-i}, Y_i \setminus \{y\}) \succcurlyeq_T (\mathbf{Z}_{-i}, Z_i \setminus \{z\})$.*

The first three axioms are *invariance properties*, in the sense that they require our poverty rankings to ignore certain aspects of the opportunity distributions and to focus on others. The first, *Anonymity,* is an axiom that requires a symmetric treatment of individuals, thereby preventing from paying attention to the identities of individuals. *Irrelevance of Inessential Opportunities* says that if the opportunity set of an individual i is reduced by the subtraction of an alternative which is not essential, then the new profile of opportunity sets exhibits the same degree of poverty as the original profile. This axiom is reminiscent of the Focus axiom, used in the income poverty paradigm, which requires invariance with respect to reduction in the incomes of the non-poor; however, instead of distinguishing between the poor and the non-poor, in the current scenario the basic distinction is between essential and non essential alternatives. *Irrelevance of Poor's Opportunity Deletions* says that if the opportunity set of a poor individual i is reduced by the subtraction of an alternative, then the new profile of opportunity sets exhibits the same degree of poverty as the original profile.

While the previous invariance properties are useful in identifyng the information that our poverty rankings should use, the next two axioms are *dominance properties*, which identify classes of transformation that have a certain effect on the poverty rankings, thereby restricting the set of poverty criteria.

The first axiom, *Dominance of essential profiles,* considers a particular case in which two profiles are composed of either empty sets or sets coinciding with the poverty treshold T. In this special case, one profiles exibith more poverty than the other if the number of empty sets in the former is higher than the number of empty sets in the latter.

Strict Monotonicity with respect to Essential Deletions says that if the opportunity set of an individual i is reduced by the subtraction of an essential alternative, then the new profile of opportunity sets exhibits a higher degree of poverty than the original profile. This axiom is a direct translation in our context of the Monotonicity axiom used in the income inequality paradigm (see Foster 2006); again the difference relies on the fact that in the current scenario the crucial distinction is between essential and non essential alternatives rather than between poor and non-poor individuals.

Finally, we propose a standard independence axiom, *Independence of Balanced Essential Deletion,* which pertains to the deletion of an essential alternative from the set of an individual i in two opportunity profiles $\mathbf{Y}, \mathbf{Z}$. Such balanced deletions preserves the ranking of the two opportunity profiles.

4. The basic characterizations

We are now able to state our characterizations of the HC and OG rankings. The first proposition characterizes the head count ranking $\succcurlyeq_T^h$.

Proposition 1. *Let $\succcurlyeq_T$ be a poverty ranking of $\mathcal{P}[X]^N$ under threshold $T \subseteq X$. Then $\succcurlyeq_T$ is the HC ranking $\succcurlyeq_T^h$ iff $\succcurlyeq_T$ satisfies* AN, IIO*,* IPOD *and* DEP*.*

Proof. It is straightforward to check that $\succcurlyeq_T^h$ is a poverty ranking and does indeed satisfy AN, IIO, DEP and IPOD.

Conversely, suppose $\succcurlyeq_T$ is a poverty ranking that satisfies AN, NT, IIO, and IPOD.

Now, consider $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$.Then, by repeated application of IIO and transitivity, $\mathbf{Y}_{|T} \succcurlyeq_T \mathbf{Z}_{|T}$.

Next, observe that $(T^{N \setminus H_T(\mathbf{Y})}, \emptyset^{H_T(\mathbf{Y})}) \sim_T \mathbf{Y}_{|T} \succcurlyeq_T \mathbf{Z}_{|T} \sim_T (T^{N \setminus H_T(\mathbf{Z})}, \emptyset^{H_T(\mathbf{Z})})$, by repeated application of IPOD. Let us now suppose that $h_T(\mathbf{Z}) > h_T(\mathbf{Y})$: then, by AN and DEP, $\mathbf{Z} \succ_T \mathbf{Y}$, a contradiction. Hence, $h_T(\mathbf{Y}) \geq h_T(\mathbf{Z})$ i.e. $\mathbf{Y} \succcurlyeq_T^h \mathbf{Z}$.

To prove the reverse inclusion, suppose that $\mathbf{Y} \succcurlyeq_T^h \mathbf{Z}$ i.e. $h_T(\mathbf{Y}) \geq h_T(\mathbf{Z})$. Then, consider $(T^{N \setminus H_T(\mathbf{Y})}, \emptyset^{H_T(\mathbf{Y})})$, $(T^{N \setminus H_T(\mathbf{Z})}, \emptyset^{H_T(\mathbf{Z})})$ and a permutation π of N such that $\pi(H_T(\mathbf{Z})) \subseteq \pi(H_T(\mathbf{Y}))$.

By IIO, $\mathbf{Y} \sim_T (T^{N \setminus H_T(\mathbf{Y})}, \emptyset^{H_T(\mathbf{Y})})$ and $\mathbf{Z} \sim_T (T^{N \setminus H_T(\mathbf{Z})}, \emptyset^{H_T(\mathbf{Z})})$; by AN, $(T^{N \setminus H_T(\mathbf{Y})}, \emptyset^{H_T(\mathbf{Y})}) \sim_T (T^{\pi(N \setminus H_T(\mathbf{Y}))}, \emptyset^{\pi(H_T(\mathbf{Y}))})$ and $(T^{N \setminus H_T(\mathbf{Z})}, \emptyset^{H_T(\mathbf{Z})}) \sim_T (T^{\pi(N \setminus H_T(\mathbf{Z}))}, \emptyset^{\pi(H_T(\mathbf{Z}))})$. Clearly, if $\pi(H_T(\mathbf{Z})) = \pi(H_T(\mathbf{Y}))$ then

$(T^{\pi(N \setminus H_T(\mathbf{Y}))}, \emptyset^{\pi(H_T(\mathbf{Y}))}) = (T^{\pi(N \setminus H_T(\mathbf{Z}))}, \emptyset^{\pi(H_T(\mathbf{Z}))})$ hence, by transitivity of $\succcurlyeq_T$, $\mathbf{Y} \sim_T \mathbf{Z}$. Let us then suppose that $\pi(H_T(\mathbf{Z})) \subset \pi(H_T(\mathbf{Y}))$. By DEP, it follows that

$(T^{\pi(N \setminus H_T(\mathbf{Y}))}, \emptyset^{\pi(H_T(\mathbf{Y}))}) \succ_T (T^{\pi(N \setminus H_T(\mathbf{Z}))}, \emptyset^{\pi(H_T(\mathbf{Z}))})$

hence in particular $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$. □

Remark 1. *The characterization provided above is tight. To check the validity of this claim, consider the following examples.*

i) To begin with, consider the non-anonymous refinement of HC defined by the following rule: $\mathbf{Y} \succcurlyeq_T^{h_1} \mathbf{Z}$ *iff i)* $\mathbf{Y} \succcurlyeq_T^{h} \mathbf{Z}$ *and* $\{Y_i, Z_i\} \subseteq \{T, \emptyset\}$ *for each* $i \in N$ *or ii)* $\mathbf{Y} \succ_T^{h} \mathbf{Z}$ *or iii)* $\mathbf{Y} \sim_T^{h} \mathbf{Z}$*, there exist* $i, j \in N$ *such* $\{Y_i, Z_j\} \cap \{T, \emptyset\} = \emptyset$*, and* $Y_1 \not\supseteq T$*. Clearly,* $\succcurlyeq_T^{h_1}$ *is a poverty ranking that satisfies IIO, IPOD and DEP, but violates AN.*

ii) Consider the refinement of HC defined by the following rule: $\mathbf{Y} \succcurlyeq_T^{h^*} \mathbf{Z}$ *iff* $\mathbf{Y} \succ_T^{h} \mathbf{Z}$ *or* $\mathbf{Y} \sim_T^{h} \mathbf{Z}$ *and* $\# \{i \in N : Y_i \supset T\} \leq \# \{i \in N : Z_i \supset T\}$*. Such a preorder is a poverty ranking that satisfies AN, DEP and IPOD but violates IIO.*

iii) Consider the universal indifference poverty ranking: i.e. $\mathbf{Y} \succcurlyeq^{I} \mathbf{Z}$ *for any* $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$*. That ranking does satisfy AN, IIO and IPOD but violates DEP.*

iv) Consider the OG-refinement of HC as defined by the following rule: $\mathbf{Y} \succcurlyeq_T^{h_g} \mathbf{Z}$ *iff either* $\mathbf{Y} \succ_T^{h} \mathbf{Z}$ *or (*$\mathbf{Y} \sim_T^{h} \mathbf{Z}$ *and* $g_T(\mathbf{Y}) \geq g_T(\mathbf{Z})$*). Such a preorder is a poverty ranking that satisfies AN, IIO and DEP, but fails to satisfy IPOD.*

Our second proposition characterizes the opportunity-gap ranking $\succcurlyeq_T^{g}$.

Proposition 2. *Let* $\succcurlyeq_T$ *be a poverty ranking of* $\mathcal{P}[X]^N$ *under threshold* $T \subseteq X$*. Then* $\succcurlyeq_T$ *is the OG ranking* $\succcurlyeq_T^{g}$ *iff* $\succcurlyeq_T$ *satisfies* AN, IIO, SMED *and* IBED*.*

Proof. It is easily checked that $\succcurlyeq_T^{g}$ is a poverty ranking and does satisfy AN, IIO, SMED and IBED.

Conversely, suppose $\succcurlyeq_T$ is a poverty ranking that satisfies AN, IIO, SMED and IBED.

Then, consider $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ such that $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$. Again, by repeated application of IIO and transitivity, $\mathbf{Y}_{|T} \succcurlyeq_T \mathbf{Z}_{|T}$. Now, suppose that $g_T(\mathbf{Z}) > g_T(\mathbf{Y})$. Then, by repeated application of IBED, $\mathbf{Z}'_{|T} \sim_T \mathbf{Y}_{|T}$ for some $\mathbf{Z}'$ such that $Z'_i \subseteq Z_i$ for each $i \in N$, and $g_T(\mathbf{Z}') = g_T(\mathbf{Y})$. It follows that, by repeated application of SMED, $\mathbf{Z}_{|T} \succ_T \mathbf{Z}'_{|T}$, hence by transitivity, $\mathbf{Z}_{|T} \succ_T \mathbf{Y}_{|T}$. Thus, by repeated application of IIO and transitivity again, $\mathbf{Z} \succ_T \mathbf{Y}$, a contradiction.

On the other hand, suppose that $\mathbf{Y} \succcurlyeq_T^{g} \mathbf{Z}$ i.e. $g_T(\mathbf{Y}) \geq g_T(\mathbf{Z})$, and consider $\mathbf{T} = (T, ..., T) \in \mathcal{P}[X]^N$. Of course, $\mathbf{T} \sim_T \mathbf{T}$, by reflexivity. Then, by AN and repeated application of IBED to $\mathbf{T} \sim_T \mathbf{T}$, it follows that $\mathbf{Y}' \succcurlyeq_T \mathbf{Z}$ for some $\mathbf{Y}'$ such that $Y'_i \setminus T = Y_i \setminus T$ and $Y_i \subseteq Y'_i$ for each $i \in N$, and $g_T(\mathbf{Y}') = g_T(\mathbf{Z})$. If in particular $g_T(\mathbf{Y}') = g_T(\mathbf{Y})$ then $\mathbf{Y}' = \mathbf{Y}$ hence $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$, and we are done. Otherwise, there exist $i \in N$ and $x \in T \cap (Y'_i \setminus Y_i)$, hence $\mathbf{Y} \succ_T \mathbf{Z}$ by transitivity and repeated application of SMED. In any case, $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$ as required. □

Remark 2. *The foregoing characterization is also tight. To verify that claim consider the following examples.*

i) Take the following non-anonymous refinement of the OG poverty ranking: $\mathbf{Y} \succcurlyeq_T^{g_1} \mathbf{Z}$ *iff* $\mathbf{Y} \succ_T^{g} \mathbf{Z}$ *or (*$\mathbf{Y} \sim_T^{g} \mathbf{Z}$*,* $Y_1 \not\supseteq T$ *and* $Z_1 \cap T \supseteq Y_1 \cap T$*). That ranking satisfies IIO, SMED and IBED but fails to satisfy AN.*

ii) Consider the following refinement of the OG poverty ranking: $\mathbf{Y} \succcurlyeq_T^{g^*} \mathbf{Z}$ *iff* $\mathbf{Y} \succ_T^{g} \mathbf{Z}$ *or (*$\mathbf{Y} \sim_T^{g} \mathbf{Z}$ *and* $\sum_{i \in N} \#(Y_i \setminus T) \leq \sum_{i \in N} \#(Z_i \setminus T)$*). That ranking satisfies AN, SMED and IBED but fails to satisfy IIO.*

iii) Consider again the universal indifference ranking: i.e. $\mathbf{Y} \succcurlyeq^{I} \mathbf{Z}$ *for any* $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$. *That preorder is a poverty ranking which does satisfy AN, IIO and IBED but violates SMED.*

iv) Consider the HC-refinement of the OG poverty ranking: $\mathbf{Y} \succcurlyeq_T^{g_h} \mathbf{Z}$ *iff* $\mathbf{Y} \succ_T^{g} \mathbf{Z}$ *or (*$\mathbf{Y} \sim_T^{g} \mathbf{Z}$ *and* $h_T(\mathbf{Y}) \geq h_T(\mathbf{Z})$*). That poverty ranking satisfies AN, IIO, SMED but violates IBED.*

5. Composite rankings

In this setion, we propose and characterize axiomatically two lexicographic rankings based on the HC and OG rankings and third one based on a linear combination of the head-count and gap criteria.

The first composite criterion, the (HG)- *lexicographic poverty ranking,* combines in a lexicographic order the HG and the OG rankings, with priority given to the HC criterion.

Definition 4. *A* (HG)- lexicographic poverty ranking *of opportunity profiles - under threshold* T*- is a binary relational system* $\left(\mathcal{P}[X]^N, \succcurlyeq_T^{hg}\right)$ *where* $\succcurlyeq_T^{hg}$ *is a preorder defined as follow: for any* $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, $\mathbf{Y} \succcurlyeq_T^{hg} \mathbf{Z}$ *if and only if either* $\mathbf{Y} \succ_T^{h} \mathbf{Z}$ *or (*$\mathbf{Y} \sim_T^{h} \mathbf{Z}$ *and* $g_T(\mathbf{Y}) \geq g_T(\mathbf{Z})$*).*

The first composite criterion, the (GH)- *lexicographic poverty ranking,* combines in a lexicographic order the HG and the OG rankings, with priority given to the OG criterion.

Definition 5. *A* (GH)- lexicographic poverty ranking *of opportunity profiles - under threshold* T*- is a binary relational system* $\left(\mathcal{P}[X]^N, \succcurlyeq_T^{gh}\right)$ *where* $\succcurlyeq_T^{gh}$ *is a preorder defined as follow: for any* $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, $\mathbf{Y} \succcurlyeq_T^{gh} \mathbf{Z}$ *if and only if either* $\mathbf{Y} \succ_T^{g} \mathbf{Z}$ *or (*$\mathbf{Y} \sim_T^{g} \mathbf{Z}$ *and* $h_T(\mathbf{Y}) \geq h_T(\mathbf{Z})$*).*

Finally, the (HG)- *weighted poverty ranking* combines the HG and the OG rankings by a linear combination.

Definition 6. *A* (HG)-weighted poverty ranking *of opportunity profiles, under threshold* T*, is a binary relational system* $\left(\mathcal{P}[X]^N, \succcurlyeq_T^{w}\right)$ *where* $\succcurlyeq_T^{w}$ *is a preorder defined as follow: there exist* w_1, $w_2 \in \mathbb{R}_{++}$ *such that, for any* $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, $\mathbf{Y} \succcurlyeq_T^{w} \mathbf{Z}$ *if and only if* $w_1 h_T(\mathbf{Y}) + w_2 g_T(\mathbf{Y}) \geq w_1 h_T(\mathbf{Z}) + w_2 g_T(\mathbf{Z})$.

5.1. More axioms.

We now propose some more axioms in order to characterize such composite rankings.

Axiom 7 (Qualified Independence of Balanced Essential Delations (Q-$IBED$))**.** *For any* $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$*, for any* $x, y, z \in X$ *and for any* $i \in N$*, such that* $Y_i \subset T$*,* $Z_i \subset T$*,* $y \in Y_i \cap T$ *and* $z \in Z_i \cap T$*:*

$$\mathbf{Y} \succcurlyeq_T \mathbf{Z} \text{ if and only if } (\mathbf{Y}_{-i}, Y_i \setminus \{y\}) \succcurlyeq_T (\mathbf{Z}_{-i}, Z_i \setminus \{z\}).$$

Axiom 8 (Conditional Dominance (CD))**.** *Let* $\succcurlyeq_T$ *be a poverty ranking with threshold* T*. Suppose there exist a positive integer* k *and* $f_1, ..., f_k \in \mathbb{R}^{\mathcal{P}[X]^N}$*, such that for all* $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$*,* $f_i(\mathbf{Y}) = f_i(\mathbf{Z})$*,* $i = 1, ..., k$ *entails* $\mathbf{Y} \sim_T \mathbf{Z}$*. Then, for all* $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$*,* $(f_1(\mathbf{Y}), ..., f_k(\mathbf{Y})) \neq (f_1(\mathbf{Z}), ..., f_k(\mathbf{Z}))$ *and* $f_i(\mathbf{Y}) \geq f_i(\mathbf{Z})$*,* $i = 1, ..., k$ *entails* $\mathbf{Y} \succ_T \mathbf{Z}$*.*

Axiom 9 (Non-Compensation (NC)). *Let $\succcurlyeq_T$ be a poverty ranking with threshold T. Suppose there exist a positive integer k and $f_1, ..., f_k \in \mathbb{R}^{\mathcal{P}[X]^N}$, such that:*

(*i*): *for all $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$: if $f_i(\mathbf{Y}) = f_i(\mathbf{Z})$, $i = 1, ..., k$, then $\mathbf{Y} \sim_T \mathbf{Z}$,*

(*ii*): *there exist $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$ and $i^* \in \{1, ..., k\}$, such that $f_{i^*}(\mathbf{Y}) > f_{i^*}(\mathbf{Z})$ and $f_j(\mathbf{Z}) > f_j(\mathbf{y})$for any $j \in \{1, ..., k\}$, $j \neq i^*$.*

Then for all $\mathbf{U}, \mathbf{V} \in \mathcal{P}[X]^N$: $\mathbf{U} \succ_T \mathbf{V}$ whenever $f_{i^*}(\mathbf{U}) > f_{i^*}(\mathbf{V})$.

Axiom 10 (Head-Count Priority (HP)). *Let $\succcurlyeq_T$ be a poverty ranking with threshold T, such that $\#T \geq 3$. For any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, if [there exist $i, j \in N$ and $x, y, z \in T$, with $x \neq y \neq z \neq x$, such that for any $l \neq i, j$, $Y_l = Z_l$, $Y_i = T \setminus \{x\}$, $Y_j = T \setminus \{y\}$, $Z_i = T$, and $Z_j = T \setminus \{x, y, z\}$],then $\mathbf{Y} \succ_T \mathbf{Z}$.*

Axiom 11 (Gap-Priority (GP)). *Let $\succcurlyeq_T$ be a poverty ranking with threshold T, such that $\#T \geq 3$. For any $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, if [there exist $i, j \in N$ and $x, y, z \in T$, with $x \neq y \neq z \neq x$, such that for any $l \neq i, j$, $Y_l = Z_l$, $Y_i = T \setminus \{x\}$, $Y_j = T \setminus \{y\}$, $Z_i = T$, and $Z_j = T \setminus \{x, y, z\}$],then $\mathbf{Z} \succ_T \mathbf{Y}$.*

Axiom 12 (Cardinal Unit-Comparability (CUC)). *Let $\succcurlyeq_T$ be a poverty ranking with threshold T. Suppose there exist a positive integer k and $f_1, ..., f_k \in \mathbb{R}^{\mathcal{P}[X]^N}$, such that for all $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$: if $f_i(\mathbf{Y}) = f_i(\mathbf{Z})$, $i = 1, ..., k$ entails $\mathbf{Y} \sim_T \mathbf{Z}$. Posit*

$$\Phi = \left\{ \begin{array}{c} \varphi = (\varphi_1, ..., \varphi_k) : \varphi_i \in \mathbb{R}^{\mathbb{R}}, i : 1, ..., k \text{ such that there exist} \\ \alpha > 0, \beta_i \in \mathbb{R} \text{ with } \varphi_i(x) = \alpha x + \beta_i \text{ for any } x \in \mathbb{R} \end{array} \right\}.$$

Then, for all $\mathbf{Y}, \mathbf{Z}, \mathbf{V}, \mathbf{U} \in \mathcal{P}[X]^N$, $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$ and $(f_1(\mathbf{U}), ..., f_k(\mathbf{U})) = ((\varphi_1 \circ f_1)(\mathbf{Y}), ..., (\varphi_k \circ f_k)(\mathbf{Y}))$ and $(f_1(\mathbf{V}), ..., f_k(\mathbf{V})) = ((\varphi_1 \circ f_1)(\mathbf{Z}), ..., (\varphi_k \circ f_k)(\mathbf{Z}))$ with $\varphi = (\varphi_1, ..., \varphi_k) \in \Phi$ entail $\mathbf{U} \succcurlyeq_T \mathbf{V}$.

The first axiom, *Qualified independence of Balanced Essential Deletion,* pertains to the deletion of an essential alternative from the set of a poor individual i in two opportunity profiles $\mathbf{Y}, \mathbf{Z}$. Such balanced deletions preserves the ranking of the two opportunity profiles. This axiom is implied by the axiom introduced before, *Independence of Balanced Essential Deletion.*

Conditional Dominance...

Non-Compensation...

The next two axioms propose two different and alternative dominance conditions, based on two basic transformation. Consider a profile $\mathbf{Y}$ and two individuals, i and j, which are just below the poverty tresholds: that is, they miss just one essential opportunity, say x and y respectively. Now cosider two different transformation: (i) a transfer of opportunity y from j to i; (ii) a decrement of one opportunity in the set available to j. By the joint effect of this double transformation, the number of individuals below the poverty tresholds has decreased - now i is not poor anymore, while j is still poor (poorer than before); however the aggregate number of opportunities that individuals i and j do not have has increased. What is the net effect on our poverty raningg? The answer will depend on the specific weight we give to the number of poor in our society vis a vis to the aggregate severity of poverty. The two axioms we introduce give different and opposite answers: according to the *Head-count priority*, poverty decreases; according to *Gap-Priority*, poverty increases.

The final two axioms set the basis for a lexicographic combination of the Head count and Opportunity Gap criteria.

Non triviality of indifference...

Cardinal Unit Comparablity...

5.2. **More results.** Before the characterization of our composite rankings, the following Lemma shows that if a ranking of opportunity profiles satisfies AN, IIO, DEP and $Q-IBED$ then, in the case in which two opportunity profiles exibith the same number of poor and the same aggregate poverty gap, they are declared to be indifferent.

Lemma 1. *Let $\succcurlyeq_T$ be a poverty ranking on $\mathcal{P}[X]^N$ and a total preorder which satisfies AN, IIO, DEP and $Q-IBED$. Then, for any $\mathbf{Y},\mathbf{Z}\in\mathcal{P}[X]^N$, $(h_T(\mathbf{Y}),g_T(\mathbf{Y}))=(h_T(\mathbf{Z}),g_T(\mathbf{Z}))$ entails $\mathbf{Y}\sim_T\mathbf{Z}$.*

Proof. Let us suppose $h_T(\mathbf{Y})=h_T(\mathbf{Z})$, $g_T(\mathbf{Y})=g_T(\mathbf{Z})$. Also, notice that for any $\mathbf{U}\in\mathcal{P}[X]^N$, $h_T(\mathbf{U})=h_T\left(\mathbf{U}_{|T}\right)$ and $g_T(\mathbf{U})=g_T\left(\mathbf{U}_{|T}\right)$ by definition of h_T and g_T respectively. Therefore, $h_T\left(\mathbf{Y}_{|T}\right)=h_T\left(\mathbf{Z}_{|T}\right)=m$ and $g_T\left(\mathbf{Y}_{|T}\right)=g_T\left(\mathbf{Z}_{|T}\right)=k$ for some m,k non-negative (observe that $m=0$ if and only if $k=0$). Next, posit $\widetilde{\mathbf{V}}=\left(\widetilde{V}\right)_{i=1,\dots,n}$ with $\widetilde{V}_i=T$ if $V_i\supseteq T$, and $\widetilde{V}_i=\emptyset$ if $V_i\nsupseteq T$ and note that $h_T\left(\mathbf{V}_{|T}\right)=h_T\left(\widetilde{\mathbf{V}}\right)$ since $\widetilde{\mathbf{V}}$ does not alter the set of poor population units in $\mathbf{V}_{|T}$. Next, $\mathbf{Y}_{|T}\succcurlyeq_T\mathbf{Z}_{|T}$ if and only if $\widetilde{\mathbf{Y}}\succcurlyeq_T\widetilde{\mathbf{Z}}$ by AN and a repeated application of $Q-IBED$ ($(m\,|T|-k)$ times). Moreover, since $h_T\left(\widetilde{\mathbf{Y}}\right)=h_T\left(\widetilde{\mathbf{Z}}\right)$ it follows by DEP that neither $\widetilde{\mathbf{Y}}\succ_T\widetilde{\mathbf{Z}}$ nor $\widetilde{\mathbf{Z}}\succ_T\widetilde{\mathbf{Y}}$. Therefore, $\widetilde{\mathbf{Y}}\sim_T\widetilde{\mathbf{Z}}$ because $\succcurlyeq_T$ is a total preorder. Finally, $\mathbf{Y}\sim_{\mathbf{T}}\mathbf{Y}_{|T}$ and $\mathbf{Z}\sim_{\mathbf{T}}\mathbf{Z}_{|T}$ by repated applications of IIO. It follows, by transitivity, that $\mathbf{Y}\sim_{\mathbf{T}}\mathbf{Z}$. □

We are now able to characterize our composite rankings. The first proposition characterizes the (HG)- *lexicographic poverty ranking* $\succcurlyeq_T^{hg}$.

Proposition 3. *Let $\succcurlyeq_T$ bea poverty ranking of $\mathcal{P}[X]^N$ under threshold $T\subseteq X$, such that $\#T\geq 3$, and a total preorder. Then, $\succcurlyeq_T=\succcurlyeq_T^{hg}$ if and only if $\succcurlyeq_T$ satisfies* AN*,* IIO*,* DEP, Q-IBED, CD, NC, *and* HP.

Proof. Let $\mathbf{Y},\mathbf{Z}\in\mathcal{P}[X]^N$, such that $\mathbf{Y}\succcurlyeq_T^{hg}\mathbf{Z}$, then one of the following cases obtains:

a) $h_T(\mathbf{Y})>h_T(\mathbf{Z})$ and $g_T(\mathbf{Y})=g_T(\mathbf{Z})$
b) $h_T(\mathbf{Y})>h_T(\mathbf{Z})$ and $g_T(\mathbf{Y})>g_T(\mathbf{Z})$
c) $h_T(\mathbf{Y})>h_T(\mathbf{Z})$ and $g_T(\mathbf{Z})>g_T(\mathbf{Y})$
d) $h_T(\mathbf{Y})=h_T(\mathbf{Z})$ and $g_T(\mathbf{Y})>g_T(\mathbf{Z})$
e) $h_T(\mathbf{Y})=h_T(\mathbf{Z})$ and $g_T(\mathbf{Y})=g_T(\mathbf{Z})$

Under case a) b), d) $\mathbf{Y}\succ_T\mathbf{Z}$ by *CD*. Under case c), $\mathbf{Y}\succ_T\mathbf{Z}$ by Lemma 1 and *NC* and *HP*. In e) by Lemma 1 $\mathbf{Y}\sim_T\mathbf{Z}$. Hence, in any case, $\mathbf{Y}\succcurlyeq_T\mathbf{Z}$.

Conversely, let $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, such that $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$, then the following cases should be distinguished:

1) $h_T(\mathbf{Y}) > h_T(\mathbf{Z})$
2) $h_T(\mathbf{Z}) > h_T(\mathbf{Y})$
3) $h_T(\mathbf{Y}) = h_T(\mathbf{Z})$ and $g_T(\mathbf{Y}) > g_T(\mathbf{Z})$
4) $h_T(\mathbf{Y}) = h_T(\mathbf{Z})$ and $g_T(\mathbf{Z}) > g_T(\mathbf{Y})$
5) $h_T(\mathbf{Y}) = h_T(\mathbf{Z})$ and $g_T(\mathbf{Z}) = g_T(\mathbf{Y})$.

Under case 1), 3), $\mathbf{Y} \succ_T^{hg} \mathbf{Z}$ by definition. Under case 2), two subcases should be distinguished, namely either $g_T(\mathbf{Z}) \geq g_T(\mathbf{Y})$ or $g_T(\mathbf{Y}) > g_T(\mathbf{Z})$. If $g_T(\mathbf{Z}) \geq g_T(\mathbf{Y})$ then by CD $\mathbf{Z} \succ_T \mathbf{Y}$, a contradiction. If, on the contrary, $g_T(\mathbf{Y}) > g_T(\mathbf{Z})$ then, by Lemma 1 and NC and HP, $\mathbf{Z} \succ_T \mathbf{Y}$ a contradiction again. Moreover, under case 4) by CD $\mathbf{Z} \succ_T \mathbf{Y}$, a contradiction. Finally, under case 5), we have that $\mathbf{Y} \sim_T^{hg} \mathbf{Z}$ by definition. Hence, the desired result follows. □

Next proposition characterizes the (GH)- *lexicographic poverty ranking* $\succcurlyeq_T^{gh}$.

Proposition 4. *Let* $\succcurlyeq_T$ *be a poverty ranking of* $\mathcal{P}[X]^N$ *under threshold* $T \subseteq X$ *such that* $\#T \geq 3$*, and a total preorder. Then,*$\succcurlyeq_T = \succcurlyeq_T^{gh}$ *if and only if* $\succcurlyeq_T$ *satisfies* AN*,* IIO*,* DEP, Q-IBED, CD, NC, *and* GP*.*

Proof. The proof replicates almost verbatim the previous one. We reproduce it here for the sake of completeness.

Let $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, such that $\mathbf{Y} \succcurlyeq_T^{gh} \mathbf{Z}$, then one of the following cases obtains:

a) $g_T(\mathbf{Y}) > g_T(\mathbf{Z})$ and $h_T(\mathbf{Y}) = h_T(\mathbf{Z})$
b) $g_T(\mathbf{Y}) > g_T(\mathbf{Z})$ and $h_T(\mathbf{Y}) > h_T(\mathbf{Z})$
c) $g_T(\mathbf{Y}) > g_T(\mathbf{Z})$ and $h_T(\mathbf{Z}) > h_T(\mathbf{Y})$
d) $g_T(\mathbf{Y}) = g_T(\mathbf{Z})$ and $h_T(\mathbf{Y}) > h_T(\mathbf{Z})$
e) $g_T(\mathbf{Y}) = g_T(\mathbf{Z})$ and $h_T(\mathbf{Y}) = h_T(\mathbf{Z})$

Under case a) b), d) $\mathbf{Y} \succ_T \mathbf{Z}$ by *CD*. Under case c), $\mathbf{Y} \succ_T \mathbf{Z}$ by Lemma 1 and *NC* and *HP*. In e) by Lemma 1 $\mathbf{Y} \sim_T \mathbf{Z}$. Hence, in any case, $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$.

Conversely, let $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, such that $\mathbf{Y} \succcurlyeq_T \mathbf{Z}$, then the following cases should be distinguished:

1) $g_T(\mathbf{Y}) > g_T(\mathbf{Z})$
2) $g_T(\mathbf{Z}) > g_T(\mathbf{Y})$
3) $g_T(\mathbf{Y}) = g_T(\mathbf{Z})$ and $h_T(\mathbf{Y}) > h_T(\mathbf{Z})$
4) $g_T(\mathbf{Y}) = g_T(\mathbf{Z})$ and $h_T(\mathbf{Z}) > h_T(\mathbf{Y})$
5) $g_T(\mathbf{Y}) = g_T(\mathbf{Z})$ and $h_T(\mathbf{Z}) = h_T(\mathbf{Y})$.

Under case 1), 3), $\mathbf{Y} \succ_T^{gh} \mathbf{Z}$ by definition. Under case 2), two subcases should be distinguished, namely either $h_T(\mathbf{Z}) \geq h_T(\mathbf{Y})$ or $h_T(\mathbf{Y}) > h_T(\mathbf{Z})$. If $h_T(\mathbf{Z}) \geq h_T(\mathbf{Y})$ then, by CD, $\mathbf{Z} \succ_T \mathbf{Y}$, a contradiction. If, on the contrary, $h_T(\mathbf{Y}) > h_T(\mathbf{Z})$ then, by Lemma 1 and NC and GP, $\mathbf{Z} \succ_T \mathbf{Y}$ a contradiction again. Moreover, under case 4), by CD, $\mathbf{Z} \succ_T \mathbf{Y}$ a contradiction. Finally, under case 5), we have that $\mathbf{Y} \sim_T^{gh} \mathbf{Z}$ by definition. Hence, the desired result. □

Our final proposition characterizes the (HG)- *weighted poverty ranking* $\succcurlyeq^w_T$.

Proposition 5. *Let* $\succcurlyeq_T$ *be a poverty ranking of* $\mathcal{P}[X]^N$ *under threshold* $T \subseteq X$ *and a total preorder. Then,* $\succcurlyeq_T = \succcurlyeq^w_T$ *if and only if* $\succcurlyeq_T$ *satisfies* AN, IIO, DEP, Q-IBED, CD, *and* CUC.

Proof. Checking that $\succcurlyeq^w_T$ is a poverty ranking which satisfies *AN, IIO, DEP, Q-IBED, CD* and *CUC* is straightforward. Then, we only need to prove the 'if' part. So, let $\mathbf{Y}, \mathbf{Z} \in \mathcal{P}[X]^N$, such that $A = (h_T(\mathbf{Y}), g_T(\mathbf{Y})) \neq (h_T(\mathbf{Z}), g_T(\mathbf{Z})) = B$ as depicted in the bidimensional Euclidean space of figure 1 below.

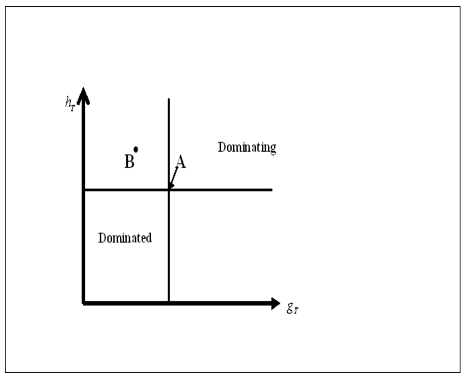

Fig.1

Now, points in the north-east corner above A represent distributions with a greater level of poverty of opportunities, while the opposite happens if we consider points of the south-west corner. Points' sets in the north-west and south-east corners outline a symmetric situation. So, let us focus on the north-west quadrant, the place of points such that $\aleph = \{\mathbf{W} : g_T(\mathbf{W}) < g_T(\mathbf{Y}), h_T(\mathbf{W}) > h_T(\mathbf{Y})\}$. If $A \succ_T B$ then we have the (GH)- *lexicographic poverty ranking* $\succcurlyeq^{gh}_T$. On the contrary, if $B \succ_T A$ we have the (HG)- *lexicographic poverty ranking* $\succcurlyeq^{hg}_T$. So, suppose $A \sim_T B$ and observe that all points lying on line joining A and B are $\sim_T$ indifferent. Indeed, $A \sim_T B$ by hypothesis. Then, $A - A \sim_T B - A$, i.e. $O \sim_T B - A$ by CUC. Hence, for any $\lambda > 0$, $O \sim_T \lambda(B - A)$ by CUC , which, in turn, entails $A \sim_T \lambda(B - A) + A$. Similarly, $O \sim_T B - A$ implies that $-(B - A) \sim_T O$. Then, for any $\lambda > 0$, $\lambda(-(B - A) \sim_T O$ entails $A + \lambda(-(B - A)) \sim_T A$. Let us denote $w_1 x + w_2 y = k$, with $w_1, w_2 \in \mathbb{R}_+$ and $k \in \mathbb{R}$ the real line joining $\mathbf{Y}$ and $\mathbf{Z}$. Moreover, observe that by CUC, $E = (h_T(\mathbf{Y}) + \delta_1, g_T(\mathbf{Y}) + \delta_2) \sim_T (h_T(\mathbf{Z}) + \delta_1, g_T(\mathbf{Z}) + \delta_2) = D$ for any $\delta_1, \delta_2 \in \mathbb{R}$. Therefore, all proper indifference curves are parallel to each other. Of course, there might exist a finite number of isolated points. But, then for each one of them, one can draw a line through it which is parallel to the other indifference curves. Finally, notice that by CD $\mathbf{U} \succ_T \mathbf{V}$ whenever $w_1 h_T(\mathbf{U}) + w_2 g_T(\mathbf{U}) = k_1$, $w_1 h_T(\mathbf{V}) + w_2 g_T(\mathbf{V}) = k_2$ and $k_1 > k_2$. □

6. Final remarks

The need for complementing the traditional evaluation of income poverty by an analysis of the deprivation suffered in many dimensions of individual and social life has been forcefully defended by many economists in the last decades. Such a measurement extension may substantially improve our understanding of the poverty in a society and may well have far-reaching policy implications. To keep the analysis as general as possible, the different dimensions have been treated in an abstract way: we have defined an opportunity set as any finite set in some arbitrary space and we have attempted to outline an axiomatic theory for the measurement of poverty of opportuinity.

We have characterized two fundamental rankings, the Head-Count and the Opportunity-Gap poverty rankings, which generalize the most widely used poverty measures used in the income poverty framework, namely the *head count ratio* and *the income poverty gap.* In addition, we have characterized axiomatically two lexicographic rankings based on the HC and OG rankings and a third one based on a linear combination of the head-count and gap criteria.

We are aware of the critique of the head count and poverty gap measures, formulated by Sen withing the income poverty framework, and based on their inability to take into account the inequality among the poor. This critique has led to the characterization of richer families of income poverty indeces (see Clarck et al. 1981 and Foster et al. 1984). It would be interesting to study such an extension in our setting: in this case we would use results obtained in the literature on opportunity inequality *and in particular...*

We have only considered comparisons of opportunity profiles for a fixed population. A possible extension of our analysis would be to compare the opportunties available to societies with different numbers of individuals. This would make it possible to rank opportunity profiles for different countries, different demographic groups, and for different time periods.

Finally, the availability of data on makes it possible an ampirical application based on the rankings characerized in this paper. This will be the object of future research.